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Universal enveloping algebra and differential calculi on inhomogeneous orthogonal q-groups

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Abstract

We review the construction of the multiparametric quantum group $ISO_{q,r}(N)$ as a projection from $SO_{q,r}(N+2)$ and show that it is a bicovariant bimodule over $SO_{q,r}(N)$. The universal enveloping algebra $U_{q,r}(iso(N))$, characterized as the Hopf algebra of regular functionals on $ISO_{q,r}(N)$, is found as a Hopf subalgebra of $U_{q,r}(so(N+2))$ and is shown to be a bicovariant bimodule over $U_{q,r}(so(N))$.

An R-matrix formulation of $U_{q,r}(iso(N))$ is given and we prove the pairing $U_{q,r}(iso(N)) \leftrightarrow ISO_{q,r}(N)$. We analyze the subspaces of $U_{q,r}(iso(N))$ that define bicovariant differential calculi on $ISO_{q,r}(N)$.

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1. Introduction

A noncommutative space–time, with a deformed Poincaré symmetry group, is an interesting geometric background for the study of Minkowski space–time physics and, in particular, of Einstein–Cartan gravity theories [9,7]. In this perspective it is natural to investigate

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inhomogeneous orthogonal quantum groups, their quantum Lie algebras and more generally their differential structure.

In this paper we review the multiparametric R-matrix formulation of $ISO_{q,r}(N)$ as a projection from $SO_{q,r}(N+2)$ [4] emphasizing the analogy with the classical construction. We also show that $ISO_{q,r}(N)$ is a bicovariant bimodule over $SO_{q,r}(N)$, freely generated by the translation elements x^a plus the dilatation element associated to $ISO_{q,r}(N)$. We then construct and analyze the universal enveloping algebra $U_{q,r}(so(N+2))$, defined as the algebra of regular functionals [11] on the multiparametric homogeneous orthogonal q-groups. The projection procedure $SO_{q,r}(N+2) \to ISO_{q,r}(N)$, initiated in [6] and developed in [7,2,4], is here exploited to obtain $U_{q,r}(iso(N))$ as a particular Hopf subalgebra of $U_{q,r}(so(N+2))$, and prove that it is paired to $ISO_{q,r}(N)$. A detailed study of $U_{q,r}(iso(N))$ and an R-matrix formulation is given. In complete analogy with the $ISO_{q,r}(N)$ case we also prove that $U_{q,r}(iso(N))$ is a bicovariant bimodule over $U_{q,r}(so(N))$ and give a basis of right invariant elements that freely generate $U_{q,r}(iso(N))$. The universal enveloping algebras of the inhomogeneous quantum groups $IGL_{q,r}(N)$, first studied with a different approach in [16], can be derived in a similar way.

The quantum Lie algebras of $ISO_{q,r}(N)$ are subspaces (adjoint submodules) of $U_{q,r}(iso(N))$, and in the last section we examine two of them, obtained as "projections" from the quantum Lie algebras of $SO_{q,r}(N+2)$. The two associated bicovariant differential calculi are also briefly presented. The first has $N+2$ generators, and is an interesting candidate for a differential calculus on the quantum orthogonal plane in dimension N. The second is obtained with the parametric restriction $r=1$; in the classical limit $r=q=1$ it reduces to the differential calculus on the undeformed $ISO(N)$. This section does not rely on the technical parts of Sections 4 and 5; these may be skipped by the reader interested mainly in the differential calculi on $ISO_{q,r}(N)$.

In this article, all the properties of the quantum inhomogeneous $ISO_{q,r}(N)$ group, its universal enveloping algebra and its differential calculus are derived from the known properties of the homogeneous "parent" structure. The main logical steps of this derivation are independent from the q-group considered, and the projection procedure may be applied to investigate more general quotients of the A, B, C, D q-groups, as for example deformed parabolic groups.

2. $SO_{q,r}(N)$ multiparametric quantum group

The $SO_{q,r}(N)$ multiparametric quantum group is freely generated by the noncommuting matrix elements $T^a{}_b$ (fundamental representation $a, b = 1, \ldots, N$) and the unit element I, modulo the relation $\det_{q,r} T = I$ and the quadratic RTT and CTT (orthogonality) relations discussed below. The noncommutativity is controlled by the R-matrix

$$R^{ab}{}_{ef} T^e{}_c T^f{}_d = T^b{}_f T^a{}_e R^{ef}{}_{cd}, \tag{2.1}$$

which satisfies the quantum Yang–Baxter equation

$$R^{a_1b_1}{}_{a_2b_2}R^{a_2c_1}{}_{a_3c_2}R^{b_2c_2}{}_{b_3c_3} = R^{b_1c_1}{}_{b_2c_2}R^{a_1c_2}{}_{a_2c_3}R^{a_2b_2}{}_{a_3b_3}, \tag{2.2}$$

a sufficient condition for the consistency of the "RTT" relations (2.1). The R-matrix components $R^{ab}{}_{cd}$ depend continuously on a (in general complex) set of parameters q_{ab}, r. For $q_{ab} = r$ we recover the uniparametric orthogonal group $SO_r(N)$ of [11]. Then $q_{ab} \to 1, r \to 1$ is the classical limit for which $R^{ab}{}_{cd} \to \delta^a_c\delta^b_d$: the matrix entries $T^a{}_b$ commute and become the usual entries of the fundamental representation. The multiparametric R-matrices for the A, B, C, D series can be found in [15] (other reference on multiparametric q-groups are given in [14,18]). For the orthogonal case they read (we use the same notations of [4]):

$$R^{ab}{}_{cd} = \delta^a_c\delta^b_d\left[\frac{r}{q_{ab}} + (r-1)\delta^{ab} + (r^{-1}-1)\delta^{ab'}\right](1-\delta^{an_2}) + \delta^a_{n_2}\delta^b_{n_2}\delta^{n_2}_c\delta^{n_2}_d$$
$$+ (r - r^{-1})[\theta^{ab}\delta^b_c\delta^a_d - \theta^{ac}r^{\rho_a-\rho_c}\delta^{a'b}\delta_{c'd}], \tag{2.3}$$

where $\theta^{ab} = 1$ for $a > b$ and $\theta^{ab} = 0$ for $a \le b$; we define $n_2 \equiv (N+1)/2$ and primed indices as $a' \equiv N + 1 - a$. The terms with the index n_2 are present only in the B_n case: $N = 2n + 1$. The ρ_a vector is given by

$$(\rho_1, \ldots, \rho_N) = \begin{cases} \left(\frac{N}{2}-1, \frac{N}{2}-2, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -\frac{N}{2}+1\right) \\ \quad \text{for } B_n[SO(2n+1)], \\ \left(\frac{N}{2}-1, \frac{N}{2}-2, \ldots, 1, 0, 0, -1, \ldots, -\frac{N}{2}+1\right) \\ \quad \text{for } D_n[SO(2n)]. \end{cases} \tag{2.4}$$

Moreover, the following relations reduce the number of independent q_{ab} parameters [15]:

$$q_{aa} = r, \quad q_{ba} = \frac{r^2}{q_{ab}}, \tag{2.5}$$

$$q_{ab} = \frac{r^2}{q_{ab'}} = \frac{r^2}{q_{a'b}} = q_{a'b'}, \tag{2.6}$$

where (2.6) also implies $q_{aa'} = r$. Therefore the q_{ab} with $a < b \le N/2$ give all the q's.

It is useful to list the nonzero complex components of the R-matrix (no sum on repeated indices):

$$\begin{aligned}
&R^{aa}{}_{aa} = r, \quad a \neq n_2, \\
&R^{aa'}{}_{aa'} = r^{-1}, \quad a \neq n_2, \\
&R^{n_2n_2}{}_{n_2n_2} = 1, \\
&R^{ab}{}_{ab} = \frac{r}{q_{ab}}, \quad a \neq b, \ a' \neq b, \\
&R^{ab}{}_{ba} = r - r^{-1}, \quad a > b, \ a' \neq b, \\
&R^{aa'}{}_{a'a} = (r - r^{-1})(1 - r^{\rho_a - \rho_{a'}}), \quad a > a', \\
&R^{aa'}{}_{bb'} = -(r - r^{-1})r^{\rho_a - \rho_b}, \quad a > b, \ a' \neq b.
\end{aligned} \tag{2.7}$$

Remark 2.1. The matrix R is upper triangular (i.e. $R^{ab}{}_{cd} = 0$ if $[a = c$ and $b < d]$ or $a < c$) and has the following properties:

$$R^{-1}_{q,r} = R_{q^{-1},r^{-1}}, \quad (R_{q,r})^{ab}{}_{cd} = (R_{q,r})^{c'd'}{}_{a'b'}, \quad (R_{q,r})^{ab}{}_{cd} = (R_{p,r})^{dc}{}_{ba}, \tag{2.8}$$

where q, r denote the set of parameters q_{ab}, r, and $p_{ab} \equiv q_{ba}$.

The inverse R^{-1} is defined by $(R^{-1})^{ab}{}_{cd}R^{cd}{}_{ef} = \delta^a_e\delta^b_f = R^{ab}{}_{cd}(R^{-1})^{cd}{}_{ef}$. The first equation in (2.8) implies that for $|q| = |r| = 1$, $\bar{R} = R^{-1}$.

Remark 2.2. The characteristic equation and the projector decomposition of $\hat{R}_{q,r}$, where $\hat{R}^{ab}{}_{cd} \equiv R^{ba}{}_{cd}$, are the same as in the uniparametric case [14,4]; in particular the projectors read:

$$\begin{aligned}
P_S &= \frac{1}{r + r^{-1}}[\hat{R} + r^{-1}I - (r^{-1} + r^{1-N})P_0],\\
P_A &= \frac{1}{r + r^{-1}}[-\hat{R} + rI - (r - r^{1-N})P_0],\\
P_0 &= (C_{ab}C^{ab})^{-1}K, \quad \text{where } K^{ab}{}_{cd} \equiv C^{ab}C_{cd}.
\end{aligned} \tag{2.9}$$

Orthogonality conditions are imposed on the elements $T^a{}_b$, consistently with the RTT relations (2.1):

$$C^{bc}T^a{}_bT^d{}_c = C^{ad}I, \qquad C_{ac}T^a{}_bT^c{}_d = C_{bd}I, \tag{2.10}$$

where the (antidiagonal) metric is

$$C_{ab} = r^{-\rho_a}\delta_{ab'} \tag{2.11}$$

and its inverse C^{ab} satisfies $C^{ab}C_{bc} = \delta^a_c = C_{cb}C^{ba}$. We see that the matrix elements of the metric and the inverse metric coincide, $C^{ab} = C_{ab}$; notice also the symmetry $C_{ab} = C_{b'a'}$.

The consistency of (2.10) with the RTT relations is due to the identities

$$C_{ab}\hat{R}^{bc}{}_{de} = (\hat{R}^{-1})^{cf}{}_{ad}C_{fe}, \tag{2.12}$$

$$\hat{R}^{bc}{}_{de}C^{ea} = C^{bf}(\hat{R}^{-1})^{ca}{}_{fd}. \tag{2.13}$$

These identities hold also for $\hat{R} \to \hat{R}^{-1}$ and can be proved using the explicit expression (2.7) of R. We also note the useful relations

$$C_{ab}\hat{R}^{ab}{}_{cd} = r^{1-N}C_{cd}, \qquad C^{cd}\hat{R}^{ab}{}_{cd} = r^{1-N}C^{ab} \tag{2.14}$$

and

$$R^{ab}{}_{cc'} = C^{ab}C_{cc'}, \quad R^{aa'}{}_{cd} = C^{aa'}C_{cd} \qquad \text{for } a > c. \tag{2.15}$$

The costructures of the orthogonal multiparametric quantum group have the same form as in the uniparametric case: the coproduct Δ, the counit ε and the coinverse κ are given by

$$\Delta(T^a{}_b) = T^a{}_b \otimes T^b{}_c, \tag{2.16}$$

$$\varepsilon(T^a{}_b) = \delta^a_b, \tag{2.17}$$

$$\kappa(T^a{}_b) = C^{ac} T^d{}_c C_{db}. \tag{2.18}$$

In order to define the quantum determinant $\det_{q,r} T$ we introduce the orthogonal N-dimensional quantum plane of coordinates x^a that satisfy the q-commutation relations $P_A{}^{ab}{}_{cd} x^c x^d = 0$. We then consider the algebra of exterior forms $\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^N$ defined by $P_S{}^{ab}{}_{cd}\, \mathrm{d}x^c\, \mathrm{d}x^d = 0$ and $P_0{}^{ab}{}_{cd}\, \mathrm{d}x^c\, \mathrm{d}x^d = 0$, i.e. (use (2.9)) $\mathrm{d}x^a\, \mathrm{d}x^b = -r R^{ba}{}_{cd}\, \mathrm{d}x^c\, \mathrm{d}x^d$. There is a natural action δ of the orthogonal quantum group on the exterior algebra (that becomes a left comodule):

$$\delta(\mathrm{d}x^a) = T^a{}_c \otimes \mathrm{d}x^c, \quad \delta(\mathrm{d}x^a\, \mathrm{d}x^b \cdots \mathrm{d}x^c) = T^a{}_d T^b{}_e \cdots T^c_f \otimes \mathrm{d}x^d\, \mathrm{d}x^e \cdots \mathrm{d}x^f.$$

Generalizing the results of [12] to the multiparametric case, we find that any N-dimensional form is proportional to the volume form $\mathrm{d}V \equiv \mathrm{d}x^1 \cdots \mathrm{d}x^N$, so that the determinant is uniquely defined by

$$\delta(\mathrm{d}V) \equiv \det_{q,r} T \otimes \mathrm{d}V. \tag{2.19}$$

Using (2.10) as in [12] it is immediate to prove that $(\det_{q,r} T)^2 = I$; moreover $\det_{q,r} T$ is central and satisfies $\Delta(\det_{q,r} T) = \det_{q,r} T \otimes \det_{q,r} T$.

To obtain the special orthogonal quantum group $SO_{q,r}(N)$ we impose also the relation $\det_{q,r} T = I$.

Remark 2.3. Using formula (2.3) or (2.7), we find that the $R^{AB}{}_{CD}$ matrix for the $SO_{q,r}(N+2)$ quantum group can be decomposed in terms of $SO_{q,r}(N)$ quantities as follows (splitting the index A as $A = (\circ, a, \bullet)$, with $a = 1, \ldots, N$):

$$R^{AB}{}_{CD} = \left(\begin{array}{c|ccccccccc}
 & \circ\circ & \circ\bullet & \bullet\circ & \bullet\bullet & \circ d & \bullet d & c\circ & c\bullet & cd \\
\circ\circ & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\circ\bullet & 0 & r^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bullet\circ & 0 & f(r) & r^{-1} & 0 & 0 & 0 & 0 & 0 & -C_{cd}\lambda r^{-\rho} \\
\bullet\bullet & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\
\circ b & 0 & 0 & 0 & 0 & \frac{r}{q_{\circ b}}\delta^b_d & 0 & 0 & 0 & 0 \\
\bullet b & 0 & 0 & 0 & 0 & 0 & \frac{r}{q_{\bullet b}}\delta^b_d & 0 & \lambda\delta^b_c & 0 \\
a\circ & 0 & 0 & 0 & 0 & \lambda\delta^a_d & 0 & \frac{r}{q_{a\circ}}\delta^a_c & 0 & 0 \\
a\bullet & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{r}{q_{a\bullet}}\delta^a_c & 0 \\
ab & 0 & -C^{ba}\lambda r^{-\rho} & 0 & 0 & 0 & 0 & 0 & 0 & R^{ab}{}_{cd}
\end{array}\right), \tag{2.20}$$

where $R^{ab}{}_{cd}$ is the R-matrix for $SO_{q,r}(N)$, C_{ab} is the corresponding metric, $\lambda \equiv r - r^{-1}$, $\rho = (N/2)$ $(r^\rho = C_{\bullet\circ})$ and $f(r) \equiv \lambda(1 - r^{-2\rho})$.

3. $ISO_{q,r}(N)$ as a projection from $SO_{q,r}(N+2)$

Classically the orthogonal group $SO(N+2)$ is defined as the set of all linear transformations with unit determinant which preserve the quadratic form $(z^0)^2 + (z^1)^2 + \cdots + (z^{N+1})^2$ or equivalently, since we are in the complex plane, the quadratic form $z^0 z^{N+1} + z^1 z^N + \cdots + z^{N+1} z^0$ (use the transformation $z^A \to (z^A + \mathrm{i} z^{A'})/\sqrt{2}$ for $A \le N/2$; $z^A \to (z^{A'} - \mathrm{i} z^A)/\sqrt{2}$ for $A > N/2$; z^A unchanged for $A = A'$). The associated metric is therefore $C_{AB} = \delta_{AB'}$ where $A, B = 0, 1, \ldots, N+1$ and $B' \equiv N + 1 - B$.

We consider the $ISO(N)$ subgroup of $SO(N+2)$ defined as follows. Select the subset of matrices in $SO(N+2)$ whose components $T^A{}_B$ read

$$T^a{}_\circ = T^\bullet{}_b = T^\bullet{}_\circ = 0. \tag{3.1}$$

The product of two such $SO(N+2)$ matrices gives a $SO(N+2)$ matrix with the same structure:

$$\begin{pmatrix} T^\circ{}_\circ & y & z \\ 0 & T & x \\ 0 & 0 & T^\bullet{}_\bullet \end{pmatrix} \cdot \begin{pmatrix} T'^\circ{}_\circ & y' & z' \\ 0 & T' & x' \\ 0 & 0 & T'^\bullet{}_\bullet \end{pmatrix} = \begin{pmatrix} T^\circ{}_\circ T'^\circ{}_\circ & y'' & z'' \\ 0 & T \cdot T' & x'' \\ 0 & 0 & T^\bullet{}_\bullet T'^\bullet{}_\bullet \end{pmatrix}, \tag{3.2}$$

where $x^c \equiv T^c{}_\bullet$, $y_a \equiv T^\circ{}_a$, $z \equiv T^\circ{}_\bullet$, $x'' = xT'^\bullet{}_\bullet + Tx'$ and $y'' = T^\circ{}_\circ y' + yT'$. These matrices form a subgroup of $SO(N+2)$. If we further set $T^\circ{}_\circ = T^\bullet{}_\bullet = 1$ this subgroup becomes $ISO(N)$.

Conditions (3.1) and $T^A{}_B \in SO(N+2)$ (i.e. $T^A{}_B C_{AC} T^C{}_D = C_{BD}$, $\det T^A{}_B = 1$) are equivalent to

$$T^a{}_\circ = T^\bullet{}_b = T^\bullet{}_\circ = 0\,, \tag{3.3}$$

$$T^a{}_b C_{ac} T^c{}_d = C_{bd}, \quad \det T^a{}_b = 1, \tag{3.4}$$

$$T^\circ{}_b = -T^a{}_b C_{ac} T^c{}_\bullet T^\circ{}_\circ, \quad T^\circ{}_\bullet = -\tfrac{1}{2} T^b{}_\bullet C_{ba} T^a{}_\bullet T^\circ{}_\circ, \quad T^\circ{}_\circ = (T^\bullet{}_\bullet)^{-1}. \tag{3.5}$$

As expected, there are no constraints on $x^c \equiv T^c{}_\bullet$.

Remark 3.1. Classically there is an easier way to recover $ISO(N)$, namely starting from $SO(N+1)$. In the quantum case the R-matrix of $SO_{q,r}(N)$ is only contained in $SO_{q,r}(N+2)$, see Remark 2.3. This explains why we have considered this bigger group.

Since $ISO(N)$ is a subgroup of $SO(N+2)$ the algebra *Fun*($ISO(N)$) of regular functions from $ISO(N)$ to $\mathbb{C}$ will be obtained from *Fun* ($SO(N+2)$) as a quotient, whose canonical projection we name P. Let us now consider the elements $T^A{}_B$ as *functions* on the $SO(N+2)$

group manifold: they define the fundamental representation of $SO(N+2)$. Since $\forall g \in ISO(N)$, $T^a{}_\circ(g) = T^\bullet{}_b(g) = T^\bullet{}_\circ(g) = 0$, we can write

$$Fun(ISO(N)) = \frac{Fun(SO(N+2))}{H}, \tag{3.6}$$

where $Fun(SO(N+2))$ is generated by $T^A{}_B$ and H is the left and right ideal generated by the functions $T^a{}_\circ$; $T^\bullet{}_b$; $T^\bullet{}_\circ$. Therefore $Fun(ISO(N))$ is generated by the functions $P(T^A{}_B)$ where P is the canonical projection associated to H: $P(T^a{}_\circ) = P(T^\bullet{}_b) = P(T^\bullet{}_\circ) = 0$; more explicitly it is generated by the elements $T^A{}_B$ modulo the relations (3.3)–(3.5).

The above construction can be carried over to the quantum group level. In this case the elements $T^A{}_B$ are abstract generators of $SO_{q,r}(N+2) \equiv Fun_{q,r}(SO(N+2))$ and we have $ISO_{q,r}(N) \equiv Fun_{q,r}(ISO(N)) = SO_{q,r}(N+2)/H$ because the ideal H is a Hopf ideal i.e.

(i) H is a two-sided ideal in $S_{q,r}(N+2)$,

(ii) H is a co-ideal, i.e.

$$\Delta_{N+2}(H) \subseteq H \otimes SO_{q,r}(N+2) + SO_{q,r}(N+2) \otimes H, \quad \varepsilon_{N+2}(H) = 0, \tag{3.7}$$

(iii) H is compatible with κ_{N+2}:

$$\kappa_{N+2}(H) \subseteq H, \tag{3.8}$$

where the suffix $N+2$ refers to the costructures of $SO_{q,r}(N+2)$. It should be clear that $ISO_{q,r}(N)$ is *not* a subalgebra, nor a Hopf subalgebra of $SO_{q,r}(N+2)$; that is why we distinguish with a suffix between the costructures of $ISO_{q,r}(N)$ and of $SO_{q,r}(N+2)$.

Following [4] the projection $P: SO_{q,r}(N+2) \to SO_{q,r}(N+2)/H$ is a Hopf algebra epimorphism, and defining the projected matrix elements $t^A{}_B = P(T^A{}_B)$, where $T^A{}_B$ are the $SO_{q,r}(N+2)$ generators, we have:

Theorem 3.1. *The quantum group $ISO_{q,r}(N)$ is generated by the matrix entries*

$$t \equiv \begin{pmatrix} P(T^\circ{}_\circ) & P(y) & P(z) \\ 0 & P(T^a{}_b) & P(x) \\ 0 & 0 & P(T^\bullet{}_\bullet) \end{pmatrix} \quad \text{and the unity } I \tag{3.9}$$

modulo the "Rtt" and "Ctt" relations

$$R^{AB}{}_{EF} t^E{}_C t^F{}_D = t^B{}_F t^A{}_E R^{EF}{}_{CD}, \tag{3.10}$$

$$C^{BC} t^A{}_B t^D{}_C = C^{AD}, \quad C_{AC} t^A{}_B t^C{}_D = C_{BD}, \tag{3.11}$$

where R and C are the multiparametric R-matrix and metric of $SO_{q,r}(N+2)$, respectively. The costructures are the same as in (2.16)–(2.18), *with $t^A{}_B$ instead of $T^a{}_b$.*

Relations (3.10) and (3.11) explicitly read:

$$R^{ab}{}_{ef}T^{e}{}_{c}T^{f}{}_{d} = T^{b}{}_{f}T^{a}{}_{e}R^{ef}{}_{cd}, \tag{3.12}$$

$$T^{a}{}_{b}C^{bc}T^{d}{}_{c} = C^{ad}I, \tag{3.13}$$

$$T^{a}{}_{b}C_{ac}T^{c}{}_{d} = C_{bd}I, \tag{3.14}$$

$$T^{b}{}_{d}x^{a} = \frac{r}{q_{d\bullet}}R^{ab}{}_{ef}x^{e}T^{f}{}_{d}, \tag{3.15}$$

$$P_{A}^{ab}{}_{cd}x^{c}x^{d} = 0, \tag{3.16}$$

$$T^{b}{}_{d}v = \frac{q_{b\bullet}}{q_{d\bullet}}vT^{b}{}_{d}, \tag{3.17}$$

$$x^{b}v = q_{b\bullet}vx^{b}, \tag{3.18}$$

$$uv = vu = I, \tag{3.19}$$

$$ux^{b} = q_{b\bullet}x^{b}u, \tag{3.20}$$

$$uT^{b}{}_{d} = \frac{q_{b\bullet}}{q_{d\bullet}}T^{b}{}_{d}u, \tag{3.21}$$

$$y_{b} = -r^{\rho}T^{a}{}_{b}C_{ac}x^{c}u, \tag{3.22}$$

$$z = -\frac{1}{(r^{-N/2}+r^{N/2-2})}x^{b}C_{ba}x^{a}u, \tag{3.23}$$

where we have set $P(T^{\circ}{}_{\circ}) = u$, $P(T^{\bullet}{}_{\bullet}) = v$ and, with abuse of notations, $T^{a}{}_{b} = P(T^{a}{}_{b})$, $x = P(x)$ $y = P(y)$, $z = P(z)$, and where $q_{a\bullet}$ are N complex parameters related by $q_{a\bullet} = r^2/q_{a'\bullet}$, with $a' = N+1-a$. The matrix P_A in Eq. (3.16) is the q-antisymmetrizer for the orthogonal quantum group, see (2.9).The last two relations (3.22) and (3.23) are constraints, showing that the $t^{A}{}_{B}$ matrix elements are really a *redundant* set. This redundance is necessary if we want an R-matrix formulation giving the q-commutations of the $ISO_{q,r}(N)$ generators. Remark that, in the R-matrix formulation for $IGL_{q,r}(N)$, *all* the $t^{A}{}_{B}$ are independent [6,2]. Here we can take as independent generators the elements

$$T^{a}{}_{b},\ x^{a},\ v,\ u \equiv v^{-1} \quad \text{and the identity } I \quad (a = 1, \ldots, N). \tag{3.24}$$

In the commutative limit $q \to 1$, $r \to 1$ we recover the algebra $Fun(ISO(N))$ described in (3.6).

Note 3.1. From the commutations (3.20) and (3.21) we see that we can set $u = I$ only when $q_{a\bullet} = 1$ for all a. From $q_{a\bullet} = r^2/q_{a'\bullet}$, cf. Eq. (2.6), this implies also $r = 1$.

Note 3.2. Eqs. (3.16) are the multiparametric orthogonal quantum plane commutations. They follow from the $({}^{a}{}_{\bullet}{}^{b}{}_{\bullet})$ Rtt components and (3.23).

Note 3.3. The u, $v = u^{-1}$ and x^a elements generate a subalgebra of $ISO_{q,r}(N)$ because their commutation relations do not involve the $T^{a}{}_{b}$ elements. Moreover these elements can be ordered using (3.16) and (3.20), and the Poincaré series of this subalgebra is the same as that of the commutative algebra in the $N+1$ symbols u, x^a [11]. A linear basis of this subalgebra is therefore given by the ordered monomials $\zeta^i = u^{i_\circ}(x^1)^{i_1}\cdots(x^N)^{i_N}$ with

$i_\circ \in \mathbb{Z}, i_1, \ldots i_N \in \mathbb{N} \cup \{0\}$. Then, using (3.15) and (3.21), a generic element of $ISO_{q,r}(N)$ can be written as $\zeta^i a_i$ where $a_i \in SO_{q,r}(N)$ and we conclude that $ISO_{q,r}(N)$ is a right $SO_{q,r}(N)$-module generated by the ordered monomials ζ^i.

One can show that as in the classical case the expressions $\zeta^i a_i$ are unique: $\zeta^i a_i = 0 \Rightarrow a_i = 0\ \forall i$, i.e. that $ISO_{q,r}(N)$ is a *free* right $SO_{q,r}(N)$-module; moreover (at least when $q_{a\bullet} = r\ \forall a$) $ISO_{q,r}(N)$ is a bicovariant bimodule over $SO_{q,r}(N)$. The proofs of these statements follow the same steps as those given after Note 5.4, and are left to the reader. Similarly, writing $a_i\zeta^i$ instead of $\zeta^i a_i$, we have that $ISO_{q,r}(N)$ is the free left $SO_{q,r}(N)$-module generated by the ζ^i.

4. Universal enveloping algebra $U_{q,r}(so(N+2))$

We construct the universal enveloping algebra $U_{q,r}(so(N+2))$ of $SO_{q,r}(N+2)$ as the algebra of regular functionals [11] on $SO_{q,r}(N+2)$.

It is the algebra over $\mathbb{C}$ generated by the counit ε and by the functionals $L^\pm$ defined by their value on the matrix elements $T^A{}_B$:

$$L^{\pm A}{}_B(T^C{}_D) = (R^\pm)^{AC}{}_{BD}, \tag{4.1}$$

$$L^{\pm A}{}_B(I) = \delta^A_B \tag{4.2}$$

with

$$(R^+)^{AC}{}_{BD} \equiv R^{CA}{}_{DB}, \quad (R^-)^{AC}{}_{BD} \equiv (R^{-1})^{AC}{}_{BD}. \tag{4.3}$$

To extend the definition (4.1) to the whole algebra $SO_{q,r}(N+2)$ we set

$$L^{\pm A}{}_B(ab) = L^{\pm A}{}_C(a) L^{\pm C}{}_B(b) \quad \forall a, b \in SO_{q,r}(N+2). \tag{4.4}$$

From (4.1), using the upper and lower triangularity of R^+ and R^-, we see that L^+ is upper triangular and L^- is lower triangular.

The commutations between $L^{\pm A}{}_B$ and $L^{\pm C}{}_D$ are induced by (2.2):

$$R_{12} L_2^\pm L_1^\pm = L_1^\pm L_2^\pm R_{12}, \tag{4.5}$$

$$R_{12} L_2^+ L_1^- = L_1^- L_2^+ R_{12}, \tag{4.6}$$

where as usual the product $L_2^\pm L_1^\pm$ is the convolution product $L_2^\pm L_1^\pm \equiv (L_2^\pm \otimes L_1^\pm)\Delta$.

The $L^{\pm A}{}_B$ elements satisfy orthogonality conditions analogous to (2.10):

$$C^{AB} L^{+C}{}_B L^{+D}{}_A = C^{DC}\varepsilon, \tag{4.7}$$

$$C_{AB} L^{\pm B}{}_C L^{\pm A}{}_D = C_{DC}\varepsilon, \tag{4.8}$$

as can be verified by applying them to the q-group generators and using (2.12) and (2.13). They provide the inverse for the matrix $L^\pm$

$$[(L^\pm)^{-1}]^A{}_B = C^{DA} L^{\pm C}{}_D C_{BC}. \tag{4.9}$$

The costructures of the algebra generated by the functionals $L^{\pm}$ and ε are defined by the duality (4.4):

$$\Delta'(L^{\pm A}{}_{B})(a\otimes b) \equiv L^{\pm A}{}_{B}(ab) = L^{\pm A}{}_{G}(a)L^{\pm G}{}_{B}(b), \tag{4.10}$$

$$\varepsilon'(L^{\pm A}{}_{B}) \equiv L^{\pm A}{}_{B}(I), \tag{4.11}$$

$$\kappa'(L^{\pm A}{}_{B})(a) \equiv L^{\pm A}{}_{B}(\kappa(a)) \tag{4.12}$$

so that

$$\Delta'(L^{\pm A}{}_{B}) = L^{\pm A}{}_{G}\otimes L^{\pm G}{}_{B}, \tag{4.13}$$

$$\varepsilon'(L^{\pm A}{}_{B}) = \delta^A_B, \tag{4.14}$$

$$\kappa'(L^{\pm A}{}_{B}) = [(L^{\pm})^{-1}]^A{}_B = C^{DA}L^{\pm C}{}_{D}C_{BC}. \tag{4.15}$$

From (4.15) we have that κ' is an inner operation in the algebra generated by the functionals $L^{\pm A}{}_{B}$ and ε; it is then easy to see that these elements generate a Hopf algebra, the Hopf algebra $U_{q,r}(so(N+2))$ of regular functionals on the quantum group $SO_{q,r}(N+2)$.

Note 4.1. From the *CLL* relations $\kappa'(L^{\pm A}{}_{B})L^{\pm B}{}_{C} = L^{\pm A}{}_{B}\kappa'(L^{\pm B}{}_{C}) = \delta^A_C\varepsilon$ we have, using upper–lower triangularity of $L^{\pm}$:

$$L^{\pm A}{}_{A}\kappa'(L^{\pm A}{}_{A}) = \kappa'(L^{\pm A}{}_{A})L^{\pm A}{}_{A} = \varepsilon, \quad \text{i.e.} \quad L^{\pm A}{}_{A}L^{\pm A'}{}_{A'} = L^{\pm A'}{}_{A'}L^{\pm A}{}_{A} = \varepsilon. \tag{4.16}$$

As a consequence $\det L^{\pm} \equiv L^{\pm\circ}{}_{\circ}L^{\pm 1}{}_{1}L^{\pm 2}{}_{2}\cdots L^{\pm N}{}_{N}L^{\pm\bullet}{}_{\bullet} = \varepsilon$. In the B_n case we also have $L^{\pm n_2}{}_{n_2} = \varepsilon$.

Note 4.2. The *RLL* relations imply that the subalgebra U^0 generated by the elements $L^{\pm A}{}_{A}$ and ε is commutative (use upper triangularity of R). Moreover, from (4.13) the invertible elements $L^{\pm A}{}_{A}$ are also group like, and we conclude that U^0 is the group Hopf algebra of the abelian group generated by $L^{\pm A}{}_{A}$ and ε. In the classical limit U^0 is a maximal commutative subgroup of $SO(N+2)$.

Note 4.3. When $q_{AB} = r$, the multiparametric R-matrix reduces to the uniparametric R-matrix and we recover the standard uniparametric orthogonal quantum groups. Then the $L^{\pm}$ functionals satisfy the further relation

$$\forall A, \quad L^{+A}{}_{A}L^{-A}{}_{A} = \varepsilon, \tag{4.17}$$

indeed $L^{+A}{}_{A}L^{-A}{}_{A}(a) = \varepsilon(a)$ as can be easily seen when $a = T^A{}_B$ and generalized to any $a\in SO_{q,r}(N+2)$ using (4.4). In this case [11] we can avoid to realize the Hopf algebra $U_r(so(N+2))$ as functionals on $SO_r(N+2)$ and we can define it abstractly as the Hopf algebra generated by the *symbols* $L^{\pm}$ and the unit ε modulo relations (4.5)–(4.8), and (4.17).

As discussed in [11] in the uniparametric case, the Hopf algebra $U_r(so(N+2))$ of regular functionals is a Hopf subalgebra of the orthogonal Drinfeld–Jimbo universal enveloping algebra U_h, where $r = e^h$. In the general multiparametric case, relation (4.17) does not hold any more. Here we discuss the generalization of (4.17) and the relation between $U_{q,r}(so(N+2))$ and the multiparametric orthogonal Drinfeld–Jimbo universal enveloping algebra $U_h^{(\mathcal{F})}$. This latter is the quasitriangular Hopf algebra $U_h^{(\mathcal{F})} = (U_h, \Delta^{(\mathcal{F})}, S, \mathcal{R}^{(\mathcal{F})})$ paired to the multiparametric orthogonal q-group $SO_{q,r}(N+2)$. It is obtained from $U_h = (U_h, \Delta, S, \mathcal{R})$ via a twist [14]. $U_h^{(\mathcal{F})}$ has the same algebra structure of U_h (and the same antipode S), while the coproduct $\Delta^{(\mathcal{F})}$ and the universal element $\mathcal{R}^{(\mathcal{F})}$ belonging to (a completion of) $U_h \otimes U_h$ are determined by the twisting element $\mathcal{F}$ that belongs to (a completion of) a maximal commutative subalgebra of $U_h \otimes U_h$. We have

$$\begin{aligned} &\forall \phi \in U_h, \quad \Delta^{(\mathcal{F})}(\phi) = \mathcal{F}\Delta(\phi)\mathcal{F}^{-1}, \\ &\mathcal{R}^{(\mathcal{F})} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \quad \mathcal{R}^{(\mathcal{F})}(T \otimes T) = R_{q,r}. \end{aligned} \tag{4.18}$$

The element $\mathcal{F}$ satisfies $(\Delta^{(\mathcal{F})} \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{13}\mathcal{F}_{23}$, $(\mathrm{id} \otimes \Delta^{(\mathcal{F})})\mathcal{F} = \mathcal{F}_{13}\mathcal{F}_{12}$, $\mathcal{F}_{12}\mathcal{F}_{21} = I$, $\mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}$, $(\varepsilon \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \varepsilon)\mathcal{F} = \varepsilon$, $(S \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes S)\mathcal{F} = \mathcal{F}^{-1}$, $\cdot(\mathrm{id} \otimes S)\mathcal{F} = \cdot(S \otimes \mathrm{id})\mathcal{F} = \cdot(\mathrm{id} \otimes \mathrm{id})\mathcal{F} = \varepsilon$; we explicitly have

$$\mathcal{F}(T^A{}_B \otimes T^C{}_D) = F^{AC}{}_{BD}, \tag{4.19}$$

where $F^{AC}{}_{BD}$ is the diagonal matrix

$$F = \mathrm{diag}\left(\sqrt{\frac{q_{11}}{r}}, \sqrt{\frac{q_{12}}{r}}, \ldots, \sqrt{\frac{q_{NN}}{r}}\right). \tag{4.20}$$

It is easy to see that the definition of the $L^{\pm}$ functionals given in the beginning of this section is equivalent to the following one: $L^{+A}{}_B(a) = \mathcal{R}^{(\mathcal{F})}(a \otimes T^A{}_B)$ and $L^{-A}{}_B(a) = \mathcal{R}^{(\mathcal{F})^{-1}}(T^A{}_B \otimes a)$. From $(\Delta^{(\mathcal{F})} \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$, $(\mathrm{id} \otimes \Delta^{(\mathcal{F})})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$, we have $\Delta^{(\mathcal{F})}(L^{\pm A}{}_B) = L^{\pm A}{}_C \otimes L^{\pm C}{}_B$ and therefore $\Delta^{(\mathcal{F})} = \Delta'$ on $U_{q,r}(so(N+2))$. From $(\mathrm{id} \otimes S)(\mathcal{R}) = (S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}$ it is also easy to see that $S = \kappa'$ on $U_{q,r}(so(N+2))$ and we conclude that the algebra of regular functionals $U_{q,r}(so(N+2))$ is a realization (in terms of functionals on $SO_{q,r}(N+2)$) of a Hopf subalgebra of $U_h^{(\mathcal{F})}$ with $r = e^h$. The generalization of (4.17) lies in $U_h^{(\mathcal{F})}$ and not in $U_{q,r}(so(N+2))$, and it is given by

$$\forall A, \quad L^{+A}{}_A L^{-A}{}_A = f_i(T^A{}_A) f^i, \quad \text{where } \mathcal{F}^4 = f_i \otimes f^i. \tag{4.21}$$

This relation holds with $L^{\pm}$ considered as abstract symbols. It can easily be checked when $L^{\pm}$ are realized as functionals: indeed $L^{+A}{}_A L^{-A}{}_A(a) = \mathcal{F}^4(T^A{}_A \otimes a)$ as can be seen when $a = T^A{}_B$ [use $\mathcal{F}^2(T^A{}_A \otimes b) = \mathcal{F}(T^A{}_A \otimes b_1)\mathcal{F}(T^A{}_A \otimes b_2)$] and generalized to any $a \in SO_{q,r}(N+2)$ using $\mathcal{F}(T^A{}_A \otimes ab) = \mathcal{F}(T^A{}_A \otimes a)\mathcal{F}(T^A{}_A \otimes b)$.

In order to characterize the relation between the Hopf algebra of regular functionals $U_{q,r}(so(N+2))$ and $U_h^{(\mathcal{F})}$, following [11], we extend the group Hopf algebra U^0 described in Note 4.2 to $\hat{U}^0$ by means of the elements [2] $l^{\pm A}{}_A = \ln L^{\pm A}{}_A$. Otherwise stated this means that in $\hat{U}^0$ we can write $L^{\pm A}{}_A = \exp(l^{\pm A}{}_A)$ where $l^{\pm A}{}_A \in \hat{U}^0$. (Explicitly $l^{\pm A}{}_A(T^C{}_D) =$

[2] In the classical limit $l^{\pm A}{}_A$ are the tangent vectors to a maximal commutative subgroup of $SO(N+2)$. They generate a Cartan subalgebra of the Lie algebra $so(N+2)$.

$\ln(R^{\pm AC}{}_{AC})\,\delta^C_D$, $l^{\pm A}{}_A(I)=0$, $l^{\pm A}{}_A(ab)=l^{\pm A}{}_A(a)\varepsilon(b)+\varepsilon(a)l^{\pm A}{}_A(b)$ and $\kappa'(l^{\pm A}{}_A)=-l^{\pm A}{}_A$.) It then follows that $\mathcal{F}$ belongs to (a completion of) $\hat{U}^0\otimes\hat{U}^0$. The corresponding extension $\hat{U}_{q,r}(so(N+2))$ of $U_{q,r}(so(N+2))$, defined as the Hopf algebra generated by the *symbols* $L^\pm$ and $l^\pm$ modulo relations (4.5)–(4.8) and (4.21), is isomorphic, when $r=\mathrm{e}^h$, to $U_h^{(\mathcal{F})}$: $\hat{U}_{q,r}(so(N+2))\cong U_h^{(\mathcal{F})}$. This relation holds because it is the twisted version of the known uniparametric analog $\hat{U}_r(so(N+2))\cong U_h$ [11,10].

The elements $L^\pm$ (or $(L^{\pm A}{}_B-\delta^A_B\varepsilon)/(r-r^{-1})$) may be seen as the quantum analog of the tangent vectors; then the RLL relations are the quantum analog of the Lie algebra relations, and we can use the orthogonal CLL conditions to reduce the number of the $L^\pm$ generators to $(N+2)(N+1)/2$, i.e. the dimension of the classical group manifold.

This we proceed to do; we next study the $RL^\pm L^\pm$ commutation relations restricted to these $(N+2)(N+1)/2$ generators and find a set of ordered monomials in the reduced $L^\pm$ that linearly span all $\hat{U}_{q,r}(so(N+2))$.

We first observe that the commutative subalgebra $\hat{U}^0$ is generated by $(N+2)/2$ elements (N even, $N=2n$) or $(N+1)/2$ elements (N odd, $N=2n+1$), for example $l^{-\circ}{}_\circ$, $l^{-1}{}_1,\ldots,l^{-n}{}_n$. For the off-diagonal $L^\pm$ elements, we can choose as free indices $(C,D)=(c,\circ)$ in relation (4.8), and using $L^{-\circ}{}_\circ L^{-\bullet}{}_\bullet=\varepsilon$, we find

$$L^{-\bullet}{}_c=-(C_{\circ\bullet})^{-1}C_{ab}L^{-b}{}_cL^{-a}{}_\circ L^{-\bullet}{}_\bullet. \tag{4.22}$$

If we choose $(C,D)=(\circ,\circ)$ we obtain

$$L^{-\bullet}{}_\circ=-(r^{-2}C_{\bullet\circ}+C_{\circ\bullet})^{-1}C_{ab}L^{-b}{}_\circ L^{-a}{}_\circ L^{-\bullet}{}_\bullet\,. \tag{4.23}$$

Similar results hold for $L^{+\circ}{}_d$ and $L^{+\circ}{}_\bullet$. Iterating this procedure, from $C_{ab}L^{-b}{}_cL^{-a}{}_d=C_{dc}\varepsilon$ we find that $L^{-N}{}_i$ (with $i=2,\ldots,N-1$) and $L^{-N}{}_1$ are functionally dependent on $L^{-i}{}_1$ and $L^{-N}{}_N$. Similarly for $L^{+1}{}_i$ and $L^{+1}{}_N$. The final result is that the elements $L^{-a}{}_J$ with $J<a<J'$ and $L^{+a}{}_J$ with $J'<a<J$ – whose number in both $\pm$ cases is $N(N+2)/4$ for N even and $(N+1)^2/4$ for N odd – and the elements $l^{-\circ}{}_\circ$, $l^{-1}{}_1,\ldots,l^{-n}{}_n$ generate all $\hat{U}_{q,r}(so(N+2))$. The total number of generators is therefore $(N+2)(N+1)/2$.

Notice that in this derivation we have not used the RLL relations (i.e. the quantum analog of the Lie algebra relations) to further reduce the number of generators. We therefore expect that, as in the classical case, monomials in the $(N+2)(N+1)/2$ generators can be ordered (in any arbitrary way). We begin by proving this for polynomials in $L^{+A}{}_A$, $L^{+\alpha}{}_J$ with $J'<\alpha<J$, and for polynomials in $L^{-A}{}_A$, $L^{-\alpha}{}_J$ with $J<\alpha<J'$.

Lemma 4.1. *Consider the $RL^\pm L^\pm$ commutation relations*

$$R^{AB}{}_{EF}L^{\pm F}{}_DL^{\pm E}{}_C=L^{\pm A}{}_EL^{\pm B}{}_FR^{EF}{}_{CD}. \tag{4.24}$$

For $C\neq D$ they close, respectively, on the subset of the $L^{+\alpha}{}_J$ with $J'<\alpha\le J$ and on the subset of the $L^{-\alpha}{}_J$ with $J\le\alpha<J'$. For $C=D$ they are equivalent to the q^{-1}-plane commutation relations

$$[P_A(J'-J+1)]^{\alpha\beta}{}_{\gamma\delta}L^{\pm\delta}{}_JL^{\pm\gamma}{}_J=0, \tag{4.25}$$

where $P_A(J'-J+1)$ is the antisymmetrizer in dimension $J-J'+1$ (compare with (2.9)) *In particular*

$$P_A{}^{ab}{}_{cd} L^{-d}{}_\circ L^{-c}{}_\circ = 0 \tag{4.26}$$

or equivalently $[(P_A)_{q^{-1},r^{-1}}]^{ab}{}_{cd}\, L^{-c}{}_\circ L^{-d}{}_\circ = 0$ *which coincide, for* $r \to r^{-1}$ *and* $q \to q^{-1}$, *with the N-dimensional quantum orthogonal plane relations* (3.16).

Proof. The proof is a straightforward calculation based on (2.15) and on upper or lower triangularity of the R-matrix and of the $L^\pm$ functionals. □

Lemma 4.2. *$U_{q,r}(so(N))$ is a Hopf subalgebra of $U_{q,r}(so(N+2))$.*

Proof. Choosing $SO_{q,r}(N)$ indices as free indices in (4.24) and using upper or lower triangularity of the $L^\pm$ matrices, and (2.7) or (2.20), we find that only $SO_{q,r}(N)$ indices appear in (4.24); similarly for relations (4.6)–(4.8), and for the costructures (4.13)–(4.15). □

Now we observe that in virtue of the RL^+L^+ relations the L^+ elements can be ordered; similarly we can order the L^- using the RL^-L^- relations. This statement can be proved by induction using that $U_{q,r}(so(N))$ is a subalgebra of $U_{q,r}(so(N+2))$, and splitting the $SO_{q,r}(N+2)$ index in the usual way (some of the resulting formulas are given in (5.9)–(5.12)).

It is then straightforward to prove that the elements $L^{+\alpha}{}_J$ with $J' < \alpha \le J$ can be ordered; indeed we can always order the $L^{+\alpha}{}_J\, L^{+\beta}{}_K$ with $J' < \alpha \le J$, $K' < \beta \le K$ and $J \ne K$ since their commutation relations are a closed subset of (4.24) (see Lemma 4.1). Then there is no difficulty in ordering substrings composed by $L^{+\alpha}{}_J$ and $L^{+\beta}{}_J$ elements because (4.25) are q^{-1}-plane commutation relations, that allow for any ordering of the quantum plane coordinates [11]. More in general the $L^{+A}{}_A$ and $L^{+\alpha}{}_J$ with $J' < \alpha < J$ can be ordered because of $L^{+A}{}_A L^{+B}{}_C = (q_{BA}/q_{CA}) L^{+B}{}_C L^{+A}{}_A$. Similarly we can order the $L^{-A}{}_A$ and $L^{-\alpha}{}_J$ with $J < \alpha < J'$. It is now easy to prove the following

Theorem 4.1. *A set of elements spanning $\hat{U}_{q,r}(so(N+2))$ is given by the ordered monomials*

$$Mon(L^{+\alpha}{}_J;\, J' < \alpha < J)\; (l^{-\circ}{}_\circ)^{p_\circ} (l^{-1}{}_1)^{p_1} \cdots (l^{-n}{}_n)^{p_n}\, Mon(L^{-\alpha}{}_J;\, J < \alpha < J'), \tag{4.27}$$

where $p_\circ, p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$, $n = N/2$ (N even), $n = (N-1)/2$ (N odd) and $Mon(L^{+\alpha}{}_J;\, J' < \alpha < J)$, $[Mon(L^{-\alpha}{}_J;\, J < \alpha < J')]$ is a monomial in the off-diagonal elements $L^{+\alpha}{}_J$ with $J' < \alpha < J$ $[L^{-\alpha}{}_J$ with $J < \alpha < J']$ where an ordering has been chosen.

Note 4.4 (*Conjecture*). The above monomials are linearly independent and therefore form a basis of $\hat{U}_{q,r}(so(N+2))$.

5. Universal enveloping algebra $U_{q,r}(iso(N))$

Consider a generic functional $f \in U_{q,r}(so(N+2))$. It is well defined on the quotient $ISO_{q,r}(N) = SO_{q,r}(N+2)/H$ if and only if $f(H)=0$. It is easy to see that the set $H^\perp$ of all these functionals is a subalgebra of $U_{q,r}(so(N+2))$: if $f(H)=0$ and $g(H)=0$ then $fg(H)=0$ because $\Delta(H) \subseteq H \otimes S_{q,r}(N+2) + S_{q,r}(N+2) \otimes H$. Moreover $H^\perp$ is a Hopf subalgebra of $U_{q,r}(so(N+2))$ since H is a Hopf ideal [19]. In agreement with these observations we will find the Hopf algebra $U_{q,r}(iso(N))$ (dually paired to $ISO_{q,r}(N)$) as a subalgebra of $U_{q,r}(so(N+2))$ vanishing on the ideal H.

Let

$$IU \equiv [L^{-A}{}_B, L^{+a}{}_b, L^{+\circ}{}_\circ, L^{+\bullet}{}_\bullet, \varepsilon] \subseteq U_{q,r}(so(N+2)) \tag{5.1}$$

be the subalgebra of $U_{q,r}(so(N+2))$ generated by $L^{-A}{}_B$, $L^{+a}{}_b$, $L^{+\circ}{}_\circ$, $L^{+\bullet}{}_\bullet$, ε.

Note 5.1. These are all and only the functionals annihilating the generators of H: $T^a{}_\circ$, $T^\bullet{}_b$ and $T^\bullet{}_\circ$. The remaining $U_{q,r}(so(N+2))$ generators $L^{+\circ}{}_b$, $L^{+a}{}_\bullet$, $L^{+\circ}{}_\bullet$ do not annihilate the generators of H and are not included in (5.1).

We now proceed to study this algebra IU. We will show that it is a Hopf algebra and that $IU \subseteq H^\perp$; we will give an R-matrix formulation, and prove that IU is a free $U_{q,r}(so(N))$-module. This is the analog of $ISO_{q,r}(N)$ being a free $SO_{q,r}(N)$-module. We then show that IU is dually paired with $ISO_{q,r}(N)$. These results lead to the conclusion that IU is the universal enveloping algebra of $ISO_{q,r}(N)$.

Theorem 5.1. *IU is a Hopf subalgebra of $U_{q,r}(so(N+2))$.*

Proof. IU is by definition a subalgebra. The sub-coalgebra property $\Delta'(IU) \subseteq IU \otimes IU$ follows immediately from the upper triangularity of $L^{+A}{}_B$:

$$\Delta'(L^{+a}{}_b) = L^{+a}{}_c \otimes L^{+c}{}_b, \qquad \Delta'(L^{+\circ}{}_\circ) = L^{+\circ}{}_\circ \otimes L^{+\circ}{}_\circ,$$
$$\Delta'(L^{+\bullet}{}_\bullet) = L^{+\bullet}{}_\bullet \otimes L^{+\bullet}{}_\bullet, \tag{5.2}$$

and the compatibility of Δ' with the product. We conclude that IU is a Hopf-subalgebra because $\kappa'(IU) \subseteq IU$ as is easily seen using (4.15) and antimultiplicativity of κ'. □

We may wonder whether the *RLL* and *CLL* relations of $U_{q,r}(so(N+2))$ close in IU. In this case IU will be given by all and *only* the polynomials in the functionals $L^{-A}{}_B$, $L^{+a}{}_b$, $L^{+\circ}{}_\circ$, $L^{+\bullet}{}_\bullet$, ε. This check is done by writing explicitly all q-commutations between the generators of IU: they do not involve the functionals $L^{+\circ}{}_b$, $L^{+a}{}_\bullet$, $L^{+\circ}{}_\bullet$. Moreover one can also write them in a compact form using a new R-matrix $\mathcal{R}_{12} \equiv \mathcal{L}^+{}_2(t_1)$, where $\mathcal{L}^+$ is defined below. Similarly the orthogonality conditions (4.7) and (4.8) do not relate elements of IU with elements not belonging to IU. We therefore conclude:

Theorem 5.2. *The Hopf algebra IU is generated by the unit ε and the matrix entries*

$$L^- = (L^{-A}{}_B), \quad \mathcal{L}^+ = \begin{pmatrix} L^{+\circ}{}_\circ & 0 & 0 \\ 0 & L^{+a}{}_b & 0 \\ 0 & 0 & L^{+\bullet}{}_\bullet \end{pmatrix}; \tag{5.3}$$

these functionals satisfy the q-commutation relations:

$$R_{12}\mathcal{L}^+{}_2\mathcal{L}^+{}_1 = \mathcal{L}^+{}_1\mathcal{L}^+{}_2R_{12} \quad \text{or equivalently} \quad \mathcal{R}_{12}\mathcal{L}^+{}_2\mathcal{L}^+{}_1 = \mathcal{L}^+{}_1\mathcal{L}^+{}_2\mathcal{R}_{12}, \tag{5.4}$$

$$R_{12}L_2^-L_1^- = L_1^-L_2^-R_{12}\,, \tag{5.5}$$

$$\mathcal{R}_{12}\mathcal{L}^+{}_2L_1^- = L_1^-\mathcal{L}^+{}_2\mathcal{R}_{12}\,, \tag{5.6}$$

where $\mathcal{R}_{12} \equiv \mathcal{L}^+{}_2(t_1)$, i.e. $\mathcal{R}^{ab}{}_{cd} = R^{ab}{}_{cd}$, $\mathcal{R}^{AB}{}_{AB} = R^{AB}{}_{AB}$ and otherwise $\mathcal{R}^{AB}{}_{CD} =$ 0,

and the orthogonality conditions:

$$C^{AB}\mathcal{L}^{+C}{}_B\mathcal{L}^{+D}{}_A = C^{DC}\varepsilon, \quad C_{AB}\mathcal{L}^{+B}{}_C\mathcal{L}^{+A}{}_D = C_{DC}\varepsilon, \tag{5.7}$$

$$C^{AB}L^{-C}{}_BL^{-D}{}_A = C^{DC}\varepsilon, \quad C_{AB}L^{-B}{}_CL^{-A}{}_D = C_{DC}\varepsilon. \tag{5.8}$$

The costructures are the ones given in (4.13)–(4.15) *with L^+ replaced by $\mathcal{L}^+$.*

Note 5.2. We can consider the extension $\hat{IU} \subset \hat{U}_{q,r}(so(N+2))$ obtained by including the elements $l^{\pm A}{}_A$ ($l^{\pm A}{}_A = \ln L^{\pm A}{}_A$, see Section 4). Then $\hat{IU}$ is generated by the symbols $L^{-A}{}_B$, $\mathcal{L}^{+A}{}_B$, $l^{\pm A}{}_A$ modulo the relations (5.4)–(5.8) and (4.21) ((4.17) in the uniparametric case). Equivalently, from (4.22) and (4.23), we have that $\hat{IU}$ is generated by $\hat{U}_{q,r}(so(N))$, the N elements $L^{-a}{}_\circ$ (satisfying the quantum plane relations) and the dilatation $l^{-\circ}{}_\circ$. All the relations are then given by those between the generators of $\hat{U}_{q,r}(so(N))$ – listed in (4.5)–(4.8), (4.21) with lower case indices – and by the following ones:

$$L^{-\circ}{}_\circ L^{-a}{}_\circ = q_{\circ a}^{-1}L^{-a}{}_\circ L^{-\circ}{}_\circ, \tag{5.9}$$

$$P_A{}^{ab}{}_{fe}L^{-e}{}_\circ L^{-f}{}_\circ = 0, \tag{5.10}$$

$$L^{-\circ}{}_\circ L^{\pm a}{}_b = \frac{q_{b\circ}}{q_{d\circ}}L^{\pm b}{}_dL^{-\circ}{}_\circ, \tag{5.11}$$

$$L^{-a}{}_\circ L^{\pm b}{}_d = \frac{r}{q_{d\circ}}(R^\pm)^{ba}{}_{ef}L^{\pm e}{}_dL^{-f}{}_\circ, \tag{5.12}$$

where $R^\pm$ is defined in (4.3). The number of generators is $N(N-1)/2 + N + 1$.

Note 5.3. When $q_{a\circ} = r\ \forall a$, then $L^{-\circ}{}_\circ = L^{+\bullet}{}_\bullet$, $L^{-\bullet}{}_\bullet = L^{+\circ}{}_\circ$ and, in complete analogy to (3.24), IU is generated by $U_{q,r}(so(N))$, $L^{-a}{}_\circ$, $L^{-\circ}{}_\circ$ and $L^{-\bullet}{}_\bullet = (L^{-\circ}{}_\circ)^{-1}$. With abuse of notations we will consider IU generated by these elements also for arbitrary values of the parameters $q_{a\circ}$; this is what actually happens in $\hat{IU}$.

Note 5.4. From the second equation in (5.4) applied to t we obtain the quantum Yang–Baxter equation for the matrix $\mathcal{R}$.

Following Note 3.3, using (5.9), (5.10) (quantum plane relations) and then (5.11) and (5.12), a generic element of IU can be written as $\eta^i a_i$ where $a_i \in U_{q,r}(so(N))$ and η^i are the ordered monomials: $\eta^i = (L^{-\circ}{}_{\circ})^{i_\circ}(L^{-1}{}_{\circ})^{i_1} \dots (L^{-N}{}_{\circ})^{i_N}$ with $i_\circ \in \mathbb{Z}, i_1, \dots, i_N \in \mathbb{N} \cup \{0\}$. Therefore IU is a right $U_{q,r}(so(N))$-module generated by the ordered monomials η^i. We now show that as in the classical case the expressions $\eta^i a_i$ are unique: $\eta^i a_i = 0 \Rightarrow a_i = 0\ \forall i$, i.e. that IU is a free right $U_{q,r}(so(N))$-module. To prove this assertion we show that, at least when $q_{a\circ} = r\ \forall a$, IU is a bicovariant bimodule over $U_{q,r}(so(N))$. Since any bicovariant bimodule is free [3] [20] we then deduce that, as a right module, IU is freely generated by the η^i.

Theorem 5.3. *Consider IU (with the parameter restriction $q_{a\circ} = r\ \forall a$) as the right $U_{q,r}(so(N))$-module $\Gamma = \{\eta^i a_i\}$ ($a_i \in U_{q,r}(so(N))$) generated by the ordered monomials $\eta^i = (L^{-\circ}{}_{\circ})^{i_\circ}(L^{-1}{}_{\circ})^{i_1} \dots (L^{-N}{}_{\circ})^{i_N}$ with $i_\circ \in \mathbb{Z}, i_1, \dots, i_N \in \mathbb{N} \cup \{0\}$.*

(a) *Γ is a bimodule with the left module structure trivially inherited from the algebra IU.*

(b) *Γ is a right covariant bimodule with right coaction $\delta_R : \Gamma \to \Gamma \otimes U_{q,r}(so(N))$ defined by*

$$\delta_R(\eta^i) \equiv \eta^i \otimes \varepsilon, \quad \delta_R(a\eta^i b) \equiv \Delta'(a)\delta_R(\eta^i)\Delta'(b). \tag{5.13}$$

(c) *Γ is a left covariant bimodule with left coaction $\delta_L : \Gamma \to U_{q,r}(so(N)) \otimes \Gamma$ defined by*

$$\delta_L(L^{-\circ}{}_{\circ}) \equiv \varepsilon \otimes L^{-\circ}{}_{\circ}, \quad \delta_L(L^{-a}{}_{\circ}) \equiv L^{-a}{}_{b} \otimes L^{-b}{}_{\circ}, \tag{5.14}$$

$$\delta_L(aL^{-\alpha}{}_{\circ}L^{-\beta}{}_{\circ} \dots L^{-\gamma}{}_{\circ}b) \equiv \Delta'(a)\delta_L(L^{-\alpha}{}_{\circ})\delta_L(L^{-\beta}{}_{\circ}) \dots \delta_L(L^{-\gamma}{}_{\circ})\Delta'(b), \tag{5.15}$$

where $\alpha = (\circ, a)$, $\beta = (\circ, b)$, $\gamma = (\circ, c)$.

(d) *Γ is a bicovariant bimodule*

$$(\mathrm{id} \otimes \delta_R)\delta_L = (\delta_L \otimes \mathrm{id})\delta_R\ . \tag{5.16}$$

(e) *Γ is freely generated by the right invariant elements η^i.*

Proof. (a) Immediate since, from Note 5.3 and Lemma 4.2, $U_{q,r}((so(N))$ is a subalgebra of IU.

(b) Consider the linear map $\delta_r : IU \to IU \otimes IU$ defined by

$$\delta_r(L^{-\alpha}{}_{\circ}) = L^{-\alpha}{}_{\circ} \otimes \varepsilon, \qquad \delta_r(a) = \Delta'(a) \quad \forall a \in U_{q,r}(so(N)) \tag{5.17}$$

and extended multiplicatively on all IU. This map is obviously well defined on $U_{q,r}(so(N))$ because it coincides with the coproduct on $U_{q,r}(so(N))$ ($U_{q,r}(so(N))$ is a Hopf subalgebra

[3] The results of [20] apply to a general Hopf algebra with invertible antipode. This can be shown by checking that all the formulae used to derive the results of [20] – they are collected in the appendix of [20] – hold also in the general case of a Hopf algebra with invertible antipode.

of IU); it is also well defined on all IU since it is multiplicative and compatible with (5.9)–(5.12). We check for example (5.12) with $q_{a\circ} = r\ \forall a$

$$\delta_r(L^{-a}{}_{\circ}L^{\pm b}{}_{d}) = L^{-a}{}_{\circ}L^{\pm b}{}_{c} \otimes L^{\pm c}{}_{d} = (R^{\pm})^{ba}{}_{ef}L^{-e}{}_{c}L^{-f}{}_{\circ} \otimes L^{\pm c}{}_{d}$$
$$= \delta_r((R^{\pm})^{ba}{}_{ef}L^{\pm e}{}_{d}L^{-f}{}_{\circ}).$$

This shows that $\delta_{\mathrm{R}} : \Gamma \to \Gamma \otimes U_{q,r}(so(N))$ is well defined since Γ is IU seen as a $U_{q,r}(so(N))$-bimodule and the actions of δ_r and δ_{R} on Γ coincide.

It is now immediate to show that Γ is a right covariant bimodule, i.e.

$$\forall \eta^i a_i \in \Gamma; \quad (\delta_{\mathrm{R}} \otimes \mathrm{id})\delta_{\mathrm{R}}(\eta^i a_i) = (\mathrm{id} \otimes \Delta')\delta_{\mathrm{R}}(\eta^i a_i),$$
$$(\mathrm{id} \otimes \varepsilon')\delta_{\mathrm{R}}(\eta^i a_i) = \eta^i a_i. \tag{5.18}$$

(c) We proceed as in the previous case, defining the linear map $\delta_{\mathrm{l}} : IU \to IU \otimes IU$,

$$\delta_{\mathrm{l}}(L^{-a}{}_{\circ}) = L^{-a}{}_{b} \otimes L^{-b}{}_{\circ}, \quad \delta_{\mathrm{l}}(L^{-\circ}{}_{\circ}) = L^{-\circ}{}_{\circ} \otimes L^{-\circ}{}_{\circ},$$
$$\delta_{\mathrm{l}}(a) = \Delta'(a) \quad \forall a \in U_{q,r}(so(N)), \tag{5.19}$$

which is extended multiplicatively on all IU. As was the case for δ_r, it is well defined on $U_{q,r}(so(N))$ and it is also well defined on all IU because it is multiplicative and compatible with (5.9)–(5.12). For example, the compatibility of δ_l with relation (5.10) holds because $P_A{}^{ab}{}_{ef}L^{-f}{}_{d}L^{-e}{}_{c} = L^{-b}{}_{f}L^{-a}{}_{e}P_A{}^{ef}{}_{cd}$ (a consequence of $(\hat{R})^{\pm 1}L_2^{\pm}L_1^{\pm} = L_2^{\pm}L_1^{\pm}(\hat{R})^{\pm 1}$ and the fact that P_A is a polynomial in $\hat{R}$ and $\hat{R}^{-1}$, see (2.9)). This is in complete analogy with the compatibility of the left coaction $\delta(x^a) = T^a{}_b \otimes x^b$ with the q-plane commutation relations.

To prove that Γ is a left covariant bimodule, notice that

$$(\varepsilon \otimes \mathrm{id})\delta_{\mathrm{l}}(L^{-a}{}_{\circ}) = L^{-a}{}_{\circ},$$
$$(\Delta' \otimes \mathrm{id})\delta_{\mathrm{l}}(L^{-a}{}_{\circ}) = L^{-a}{}_{d} \otimes L^{-d}{}_{b} \otimes L^{-b}{}_{\circ} = (\mathrm{id} \otimes \delta_{\mathrm{l}})\delta_{\mathrm{l}}(L^{-a}{}_{\circ}), \tag{5.20}$$

and similarly for $L^{-\circ}{}_{\circ}$. Now since $\delta_{\mathrm{l}}(a) = \Delta'(a)$ if $a \in U_r(so(N))$, and since δ_{l} is multiplicative, we have on all IU

$$(\varepsilon \otimes \mathrm{id})\delta_{\mathrm{l}} = \mathrm{id}, \quad (\Delta' \otimes \mathrm{id})\delta_{\mathrm{l}} = (\mathrm{id} \otimes \delta_{\mathrm{l}})\delta_{\mathrm{l}}. \tag{5.21}$$

(d) The bicovariance condition (5.16) follows directly from

$$(\mathrm{id} \otimes \delta_r)\delta_{\mathrm{l}}(L^{-a}{}_{\circ}) = L^{-a}{}_{b} \otimes L^{-b}{}_{\circ} \otimes \varepsilon = (\delta_{\mathrm{l}} \otimes \mathrm{id})\delta_r(L^{-a}{}_{\circ}), \tag{5.22}$$

$$(\mathrm{id} \otimes \delta_r)\delta_{\mathrm{l}}(L^{-\circ}{}_{\circ}) = \varepsilon \otimes L^{-\circ}{}_{\circ} \otimes \varepsilon = (\delta_{\mathrm{l}} \otimes \mathrm{id})\delta_r(L^{-\circ}{}_{\circ}). \tag{5.23}$$

(e) We now recall that a bicovariant bimodule is always freely generated by a basis of Γ_{inv}, the space of right invariant elements of Γ [20]. We also know that the η^i are right invariant. Now, since they generate Γ, they linearly span Γ_{inv}, and since they are linearly independent, they form a basis of Γ_{inv}. We conclude that Γ is freely generated by the η^i: $\eta^i a_i = 0 \Rightarrow a_i = 0\ \forall i$. □

It is now easy to prove that the η^i freely generate IU also without the restriction $q_{a\circ} = r\ \forall a$. (Hint: recall the definition of L^- as $L^{-A}{}_{B}(c) = \mathcal{R}^{(\mathcal{F})^{-1}}(T^A{}_B \otimes c)\ \forall c \in SO_{q,r}(N+2)$, and use $\mathcal{F} \in \hat{U}^0 \otimes \hat{U}^0$ to show that $L^{-A}{}_B$ differs from the uniparametric $L^{-A}{}_B$ (obtained with $\mathcal{R}$ instead of $\mathcal{R}^{(\mathcal{F})}$) by a factor belonging to $\hat{U}^0$ and invertible.)

5.1. Duality $U_{q,r}(iso(N)) \leftrightarrow ISO_{q,r}(N)$

We now show that IU is dually paired to $SO_{q,r}(N+2)$. This is the fundamental step allowing to interpret IU as the algebra of regular functionals on $ISO_{q,r}(N)$.

Theorem 5.4. *IU annihilates H, i.e. $IU \subseteq H^{\perp}$.*

Proof. Let $\mathcal{L}$ and $\mathcal{T}$ be generic generators of IU and H, respectively. As discussed in Note 5.1, $\mathcal{L}(\mathcal{T}) = 0$. A generic element of the ideal is given by $a\mathcal{T}b$ where sum of polynomials is understood; we have (using Sweedler's notation for the coproduct): $\mathcal{L}(a\mathcal{T}b) = \mathcal{L}_{(1)}(a)\mathcal{L}_{(2)}(\mathcal{T})\mathcal{L}_{(3)}(b) = 0$ because $\mathcal{L}_{(2)}(\mathcal{T}) = 0$. Indeed $\mathcal{L}_{(2)}$ is still a generator of IU since IU is a sub-coalgebra of $U_{q,r}(so(N+2))$. Thus $\mathcal{L}(H) = 0$. Recalling that a product of functionals annihilating H still annihilates the co-ideal H, we also have $IU(H) = 0$. □

In virtue of Theorem 5.4 the following bracket is well defined:

$$\langle , \rangle : \begin{array}{l} IU \otimes ISO_{q,r}(N) \longrightarrow \mathbb{C} \\ \langle a', P(a)\rangle \equiv a'(a) \quad \forall a' \in IU, \quad \forall a \in SO_{q,r}(N+2), \end{array} \tag{5.24}$$

where $P : SO_{q,r}(N+2) \to SO_{q,r}(N+2)/H \equiv ISO_{q,r}(N)$ is the canonical projection, which is surjective. The bracket is well defined because two generic counterimages of $P(a)$ differ by an addend belonging to H.

Note that when we use the bracket $\langle\ ,\ \rangle$, a' is seen as an element of IU, while in the expression $a'(a)$, a' is seen as an element of $U_{q,r}(so(N+2))$ (vanishing on H).

Theorem 5.5. *The bracket* (5.24) *defines a pairing between IU and $ISO_{q,r}(N)$: $\forall a', b' \in IU$, $\forall P(a), P(b) \in ISO_{q,r}(N)$*

$$\langle a'b', P(a)\rangle = \langle a' \otimes b', \Delta(P(a))\rangle, \tag{5.25}$$

$$\langle a', P(a)P(b)\rangle = \langle \Delta'(a'), P(a) \otimes P(b)\rangle, \tag{5.26}$$

$$\langle \kappa'(a'), P(a)\rangle = \langle a', \kappa(P(a))\rangle, \tag{5.27}$$

$$\langle I, P(a)\rangle = \varepsilon(P(a)), \quad \langle a', P(I)\rangle = \varepsilon'(a'). \tag{5.28}$$

Proof. The proof is easy since IU is a Hopf subalgebra of $U_{q,r}(so(N+2))$ and P is compatible with the structures and costructures of $SO_{q,r}(N+2)$ and $ISO_{q,r}(N)$. Indeed we have

$$\begin{aligned} \langle a', P(a)P(b)\rangle &= \langle a', P(ab)\rangle = a'(ab) \\ &= \Delta'(a')(a \otimes b) = \langle \Delta'(a'), P(a) \otimes P(b)\rangle, \\ \langle a'b', P(a)\rangle &= a'b'(a) = (a' \otimes b')\Delta_{N+2}(a) \\ &= \langle a' \otimes b', (P \otimes P)\Delta_{N+2}(a)\rangle = \langle a' \otimes b', \Delta(P(a))\rangle, \end{aligned}$$

$$\langle \kappa'(a'), P(a)\rangle = \kappa'(a')(a) = a'(\kappa_{N+2}(a))$$
$$= \langle a', P(\kappa_{N+2}(a))\rangle = \langle a', \kappa(P(a))\rangle. \qquad \square$$

We now recall that IU and $ISO_{q,r}(N)$, besides being dually paired, are free right modules, respectively, on $U_{q,r}(so(N))$ and on $SO_{q,r}(N)$. They are freely generated by the two isomorphic sets of the ordered monomials in $L^{-\circ}{}_{\circ}$, $L^{-a}{}_{\circ}$ and u, x^a, respectively (cf. the commutations (5.9), (5.10) and (3.20), (3.16)). We can then call IU the universal enveloping algebra of $ISO_{q,r}(N)$

$$U_{q,r}(iso(N)) \equiv IU \tag{5.29}$$

in the same way $U_r(so(N))$ is the universal enveloping algebra of $SO_r(N)$ [11].

Note 5.5. Given a $*$-structure on $ISO_{q,r}(N)$, the duality $ISO_{q,r}(N) \leftrightarrow U_{q,r}(iso(N))$ induces a $*$-structure on $U_{q,r}(iso(N))$. If in particular the $*$-conjugation on $ISO_{q,r}(N)$ is found by projecting a $*$-conjugation on $SO_{q,r}(N+2)$, then the induced $*$ on $U_{q,r}(iso(N))$ is simply the restriction to $U_{q,r}(iso(N))$ of the $*$ on $U_{q,r}(so(N+2))$. This is the case for the $*$-structures that lead to the real forms $ISO_{q,r}(N, \mathbb{R})$ and $ISO_{q,r}(n+1, n-1)$ and in particular to the quantum Poincaré group [8,7,4].

6. Projected differential calculus

In the previous sections we have found the inhomogeneous quantum group $ISO_{q,r}(N)$ by means of a projection from $SO_{q,r}(N+2)$. Dually, its universal enveloping algebra is a given Hopf subalgebra of $U_{q,r}(so(N+2))$. Using the same techniques differential calculi on $ISO_{q,r}(N)$ can be found.

6.1. Projecting Woronowicz ideal

Following Woronowicz [20], we recall that a bicovariant differential calculus over a generic Hopf algebra A is determined by a right ideal R of A. This ideal has to be included in $\ker \varepsilon$ (i.e. its elements have vanishing counit) and must be ad-invariant, that is, $ad_A(r) \in R \otimes A \, \forall r \in R$ where $ad_A(r)$ is defined by $ad_A(a) \equiv a_2 \otimes \kappa_A(a_1)a_3 \, \forall a \in A$; the index A denotes the costructures in A and we have used Sweedler's notation for the coproduct. For any such R one can construct a bicovariant differential calculus. In the following we show that, given a quotient Hopf algebra A/H (with canonical projection $P : A \to A/H \equiv P(A)$), $P(R)$ is a right ad-invariant ideal in $P(A)$; therefore it defines a bicovariant differential calculus at the projected level. Moreover the space of tangent vectors on $P(A)$ is easily found as a subspace of the tangent vectors on A. The explicit construction of the exterior differential d, and of the bicovariant bimodule Γ of one-forms is then straightforward.

Theorem 6.1. *If $R \in \ker \varepsilon$ is a right ad-invariant ideal of A then $P(R)$ is included in $\ker \varepsilon$ and is a right ad-invariant ideal of $P(A)$.*

Proof. The only nontrivial part is ad-invariance. From $ad_A(r) = r_2 \otimes \kappa_A(r_1)r_3 \in R \otimes A$ $\forall r \in R$, applying $P \otimes P$ we obtain $P(r_2) \otimes P(\kappa_A(r_1))P(r_3) \in P(R) \otimes P(A)$ $\forall P(r) \in P(R)$. Now

$$\begin{aligned} P(r_2) \otimes P(\kappa_A(r_1))P(r_3) &= P(r_2) \otimes \kappa(P(r_1))P(r_3) \\ &= P(r)_2 \otimes \kappa(P(r)_1)P(r)_3 \equiv ad(P(r)), \end{aligned} \tag{6.1}$$

where we have used compatibility of the projection with the costructures of A and $P(A)$; κ denotes the antipode in $P(A)$ and, after the second equality, the coproduct of $P(A)$ is understood. Relation (6.1) gives the ad-invariance of $P(R)$: $\forall P(r) \in P(R)$, $ad(P(r)) \in P(R) \otimes P(A)$. □

The space of tangent vectors on a quantum group $P(A)$ is given by [20]

$$T \equiv \{\bar{\chi} : P(A) \to \mathbb{C} \,|\, \bar{\chi} \text{linear functionals}, \bar{\chi}(I) = 0 \text{ and } \bar{\chi}(P(R)) = 0\}\,. \tag{6.2}$$

Remark 6.1. The vector space T defined in (6.2) is given by all and only those functionals $\bar{\chi}$ corresponding to elements χ of the tangent space T_A on A that annihilate the Hopf ideal H. Indeed if χ annihilates H, then $\bar{\chi}$ defined by $\bar{\chi} : A/H \longrightarrow \mathbb{C}$ with $\bar{\chi}(P(a)) \equiv \chi(a)$, $\forall P(a) \in P(A)$, is a well-defined functional on $P(A)$ (see (5.24)). From $\chi(R) = 0$ we obtain $\bar{\chi}(P(R)) = 0$, i.e. $\bar{\chi} \in T$. Vice versa a functional $\bar{\chi} \in T$ is trivially extended to a functional $\chi \in T_A$.

Recall [20,17] that the deformed Lie bracket is given by $[\chi_i, \chi_j](a) = (\chi_i \otimes \chi_j)ad_A(a)$ where χ_i, χ_j are functionals on A. For the "projected" q-Lie algebra we have:

Theorem 6.2. *The q-Lie algebra on $P(A)$ is a closed subset of the q-Lie algebra on A.*

Proof. Let $\chi_i(H) = \chi_j(H) = 0$. We have, using (6.1) in the second equality

$$\begin{aligned} [\bar{\chi}_i, \bar{\chi}_j](P(a)) &= (\bar{\chi}_i \otimes \bar{\chi}_j)ad(P(a)) = \bar{\chi}_i \otimes \bar{\chi}_j(P \otimes P)ad_A(a) \\ &= (\chi_i \otimes \chi_j)ad_A(a) = [\chi_i, \chi_j](a), \end{aligned}$$

in particular $[\bar{\chi}_i, \bar{\chi}_j](P(R)) = [\chi_i, \chi_j](R) = 0$ and this proves the theorem. □

In virtue of Theorem 6.2 the following corollary is easily proved.

Corollary 6.1. *Consider the structure constants ${\mathbb{C}_{ij}}^k$ defined by $[\chi_i, \chi_j] = {\mathbb{C}_{ij}}^k \chi_k$, where $\{\chi_i\}$ will henceforth denote a basis of T_A containing the maximum number of tangent vectors vanishing on H. The subset of the structure constants corresponding to the functionals χ_i that annihilate H is the set of all the structure constants of $P(A)$.*

The exterior differential related to this projected calculus is given by

$$\forall a \in P(A), \quad \mathrm{d}a = (\bar{\chi}_i * a)\bar{\omega}^i, \tag{6.3}$$

where $\bar{\chi}_i * a \equiv (\mathrm{id} \otimes \chi_i)\Delta a$, and $\bar{\omega}^i$ are the one-forms dual to the tangent vectors $\bar{\chi}_i$ [20,5]; they freely generate the left module of one-forms $\Gamma = \{a_i \bar{\omega}^i, a_i \in P(A)\}$. The right module structure is given by the $\bar{f}^i{}_j$ functionals, obtained applying the coproduct Δ' to the $\bar{\chi}_i$

$$\Delta' \bar{\chi}_i = \bar{\chi}_j \otimes \bar{f}^j{}_i + \varepsilon \otimes \bar{\chi}_i \Rightarrow \bar{\omega}^i a = (\bar{f}^i{}_j * a)\bar{\omega}^j \ . \tag{6.4}$$

The space Γ of one-forms on $P(A)$ can be studied by projecting the one-forms on A into one-forms on $P(A)$. For this we introduce the projection P acting on Γ_A (the space of one-forms on A) as follows:

$$P : \Gamma_A \to \Gamma \tag{6.5}$$

$$a_i \omega^i \mapsto P(a_i)\bar{\omega}^i, \tag{6.6}$$

where $\bar{\omega}^i = 0$ if $\chi_i(H) \neq 0$. We now show that P is a bicovariant bimodule epimorphism and that it is compatible with the differential calculi. Trivially P is a left module epimorphism because Γ_A and Γ are free left modules generated respectively by the one-forms $\{\omega^i\}$ and $\{\bar{\omega}^i\}$. It is also easy to see (use (6.4)) that $\forall \rho \in \Gamma_A, \forall a \in A$, $P(\rho a) = P(\rho)P(a)$, which shows that P is a bimodule epimorphism.

To prove that P is compatible with the exterior differentials d_A on A and d on $P(A)$, consider the generic one-form $a\, \mathrm{d}_A b = a(\chi_i * b)\omega^i$ (see (6.3)); we have $P(a \mathrm{d}_A b) = P(a)P(\chi_i * b)\bar{\omega}^i = P(a)[\bar{\chi}_i * P(b)]\bar{\omega}^i = P(a)\, \mathrm{d}P(b)$.

The exterior differential d induces the comodule structure on Γ by the definitions:

$$\forall a, b \in P(A) \qquad \begin{aligned} \Delta_{\mathrm{L}}(a\, \mathrm{d}b) &\equiv \Delta(a)(\mathrm{id} \otimes \mathrm{d})\Delta(b), \\ \Delta_{\mathrm{R}}(a\, \mathrm{d}b) &\equiv \Delta(a)(\mathrm{d} \otimes \mathrm{id})\Delta(b). \end{aligned} \tag{6.7}$$

Finally P is a comodule homomorphism: $\Delta_{\mathrm{L}}(P(\rho)) = (P \otimes P)\Delta_{\mathrm{LA}}(\rho)$; $\Delta_{\mathrm{R}}(P(\rho)) = (P \otimes P)\Delta_{\mathrm{RA}}(\rho)$, $\forall \rho \in \Gamma_A$ where Δ_{LA} (Δ_{RA}) is the left (right) coaction of A.

From $\Delta_{\mathrm{LA}}\omega^i = I \otimes \omega^i$ and $\Delta_{\mathrm{RA}}\omega^i = \omega^j \otimes M_j{}^i$, where $M_j{}^i$ defines the adjoint representation on A, we obtain an explicit expression for Δ_{L} and Δ_{R},

$$\Delta_{\mathrm{L}}\bar{\omega}^i = I \otimes \bar{\omega}^i, \qquad \Delta_{\mathrm{R}}\bar{\omega}^i = \bar{\omega}^j \otimes P(M_j{}^i) \ . \tag{6.8}$$

6.2. Application: $ISO_{q,r}(N)$ differential calculi

We now apply the above discussion to the quantum groups $A = SO_{q,r}(N+2)$ and $P(A) = ISO_{q,r}(N)$. The adjoint representation for $SO_{q,r}(N+2)$ is given by

$$M^{A\ D}_{BC} \equiv T^A{}_C \kappa_{N+2}(T^D{}_B) \ , \tag{6.9}$$

and the χ functionals explicitly read

$$\chi^A{}_B = \frac{1}{r - r^{-1}}[f_C{}^{CA}{}_B - \delta^A_B \varepsilon], \quad \text{where } f_{A_1}{}^{A_2 B_1}{}_{B_2} \equiv \kappa'(L^{+B_1}{}_{A_1})L^{-A_2}{}_{B_2}, \tag{6.10}$$

see [13] and references therein (see also [3]). Decomposing the indices we find:

$$\chi^a_{\ b} = \frac{1}{r-r^{-1}}[f_c{}^{ca}{}_b - \delta^a_b \varepsilon] \quad + \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet a}{}_b, \tag{6.11}$$

$$\chi^a_{\ \circ} = \frac{1}{r-r^{-1}} f_c{}^{ca}{}_\circ \quad + \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet a}{}_\circ, \tag{6.12}$$

$$\chi^\circ_{\ b} = \quad + \frac{1}{r-r^{-1}}[f_c{}^{c\circ}{}_b + f_\bullet{}^{\bullet\circ}{}_b], \tag{6.13}$$

$$\chi^a_{\ \bullet} = \quad + \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet a}{}_\bullet, \tag{6.14}$$

$$\chi^\bullet_{\ b} = \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet\bullet}{}_b, \tag{6.15}$$

$$\chi^\circ_{\ \circ} = \frac{1}{r-r^{-1}}[f_\circ{}^{\circ\circ}{}_\circ - \varepsilon] \quad + \frac{1}{r-r^{-1}}[f_c{}^{c\circ}{}_\circ + f_\bullet{}^{\bullet\circ}{}_\circ], \tag{6.16}$$

$$\chi^\circ_{\ \bullet} = \quad + \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet\circ}{}_\bullet, \tag{6.17}$$

$$\chi^\bullet_{\ \circ} = \frac{1}{r-r^{-1}} f_\bullet{}^{\bullet\bullet}{}_\circ, \tag{6.18}$$

$$\chi^\bullet_{\ \bullet} = \underbrace{\frac{1}{r-r^{-1}}[f_\bullet{}^{\bullet\bullet}{}_\bullet - \varepsilon],}_{\text{terms annihilating } H} \tag{6.19}$$

where using Theorem 5.4 and Note 5.1 we have indicated the terms that do and do not annihilate the Hopf ideal H. We see that only three of these functionals, namely $\chi^\bullet_{\ b}$, $\chi^\bullet_{\ \circ}$ and $\chi^\bullet_{\ \bullet}$, do vanish on H. The resulting bicovariant differential calculus contains dilatations and translations, but does not contain the tangent vectors of $SO_{q,r}(N)$, i.e. the functionals $\chi^a_{\ b}$. The differential related to this calculus is given by

$$\forall a \in ISO_{q,r}(N) \quad da = (\chi^\bullet_{\ b} * a)\omega_\bullet{}^b + (\chi^\bullet_{\ \bullet} * a)\omega_\bullet{}^\bullet + (\chi^\bullet_{\ \circ} * a)\omega_\bullet{}^\circ, \tag{6.20}$$

where $\omega_\bullet{}^b$, $\omega_\bullet{}^\bullet$ and $\omega_\bullet{}^\circ$ are the one-forms dual to the tangent vectors $\chi^\bullet_{\ b}$, $\chi^\bullet_{\ \circ}$ and $\chi^\bullet_{\ \bullet}$ [20,5] (with abuse of notation, we omit the bar over the elements of the projected calculus). The q-Lie algebra is explicitly given by [4]

$$\chi^\bullet_{\ \circ}\chi^\bullet_{\ b} - (q_{\bullet b})^{-2}\chi^\bullet_{\ b}\chi^\bullet_{\ \circ} = 0, \tag{6.21}$$

$$\chi^\bullet_{\ c}\chi^\bullet_{\ \bullet} - r^{-2}\chi^\bullet_{\ \bullet}\chi^\bullet_{\ c} = -r^{-1}\chi^\bullet_{\ c}, \tag{6.22}$$

$$\chi^\bullet_{\ \circ}\chi^\bullet_{\ \bullet} - r^{-4}\chi^\bullet_{\ \bullet}\chi^\bullet_{\ \circ} = \frac{-(1+r^2)}{r^3}\chi^\bullet_{\ \circ}, \tag{6.23}$$

$$q_{\bullet a} P_A{}^{ab}{}_{cd}\chi^\bullet_{\ b}\chi^\bullet_{\ a} = 0. \tag{6.24}$$

A combination of the above relations yields

$$\chi^\bullet_{\ \circ} + \lambda\chi^\bullet_{\ \circ}\chi^\bullet_{\ \bullet} = \lambda\frac{-r^{N/2}}{r^2+r^N}\frac{1}{q_{d\bullet}}\chi^\bullet_{\ b}C^{db}\chi^\bullet_{\ d}. \tag{6.25}$$

[4] We thank A. Scarfone for the derivation of (6.24).

Notice the similar structure of Eqs. (3.23), (4.23) and (6.25).

The bicovariant bimodule of one-forms is characterized by the functionals

$$f_{\bullet}{}^{\circ\bullet}{}_{\circ},\ f_{\bullet}{}^{a\bullet}{}_{\circ},\ f_{\bullet}{}^{\bullet\bullet}{}_{\circ},\ f_{\bullet}{}^{a\bullet}{}_{b},\ f_{\bullet}{}^{\bullet\bullet}{}_{b},\ f_{\bullet}{}^{\bullet\bullet}{}_{\bullet} \tag{6.26}$$

that appear in the comultiplication of $\chi^{\bullet}_{\ b}$, $\chi^{\bullet}_{\ \circ}$ and $\chi^{\bullet}_{\ \bullet}$ [use upper (lower) triangularity of L^{+} (L^{-})], and by the elements

$$P(M^{\bullet}{}_{B\bullet}{}^{D}) = P(T^{\bullet}{}_{\bullet}\kappa_{N+2}(T^{D}{}_{B})) = vP(\kappa_{N+2}(T^{D}{}_{B})) \tag{6.27}$$

that, according to (6.9) and (6.8), characterize the right coaction of $ISO_{q,r}(N)$ on $\omega_{\bullet}{}^{b}$, $\omega_{\bullet}{}^{\bullet}$ and $\omega_{\bullet}{}^{\circ}$. They explicitly read

$$P(M^{\bullet}{}_{\circ\bullet}{}^{\circ}) = v^{2},\quad P(M^{\bullet}{}_{\circ\bullet}{}^{d}) = 0,\quad P(M^{\bullet}{}_{\circ\bullet}{}^{\bullet}) = 0,$$

$$P(M^{\bullet}{}_{b\bullet}{}^{\circ}) = vr^{-N/2}x^{e}C_{eb},\quad P(M^{\bullet}{}_{b\bullet}{}^{d}) = v\kappa(T^{d}{}_{b}),\quad P(M^{\bullet}{}_{b\bullet}{}^{\bullet}) = 0,$$

$$P(M^{\bullet}{}_{\bullet\bullet}{}^{\circ}) = -\frac{1}{r^{N}(r^{N/2}+r^{-N/2+2})}x^{e}C_{ef}x^{f},\ P(M^{\bullet}{}_{\bullet\bullet}{}^{d}) = v\kappa(x^{d}),\ P(M^{\bullet}{}_{\bullet\bullet}{}^{\bullet}) = I. \tag{6.28}$$

Notice that only the couples of indices $(\bullet\circ)$, $(\bullet b)$ and $(\bullet\bullet)$ appear in (6.20)–(6.28): these are therefore the only indices involved in the projected differential calculus on $ISO_{q,r}(N)$.

The functionals $\chi^{a}_{\ b}$ cannot be good tangent vectors on $ISO_{q,r}(N)$ because of the functionals $f_{\bullet}{}^{\bullet a}{}_{b}$ appearing in (6.11): these do not annihilate H. We see however that $\lim_{r\to 1}$ $(1/(r-r^{-1}))f_{\bullet}{}^{\bullet a}{}_{b}(a) = 0, \forall a \in SO_{q,r}(N+2)$; for this reason we consider in the following the particular multiparametric deformations called "minimal deformations" (twistings), corresponding to $r = 1$.

As shown in [3] in the $r \to 1$ limit the χ functionals are given by

$$\chi^{A}_{\ A} = \lim_{r\to 1}\frac{1}{\lambda}\,[f_{A}{}^{AA}{}_{A} - \varepsilon],\qquad \chi^{A}_{\ A'} = 0,$$

$$\chi^{A}_{\ B} = \lim_{r\to 1}\frac{1}{\lambda}f_{A}{}^{AA}{}_{B},\quad A > B,\qquad \chi^{A}_{\ B} = \lim_{r\to 1}\frac{1}{\lambda}f_{B}{}^{BA}{}_{B},\quad A < B,$$

where $\lambda \equiv r - r^{-1}$, and close on the q-Lie algebra

$$\begin{aligned}
&\chi^{C_1}_{\ C_2}\chi^{B_1}_{\ B_2} - q_{B_1C_2}q_{C_1B_1}q_{B_2C_1}q_{C_2B_2}\chi^{B_1}_{\ B_2}\chi^{C_1}_{\ C_2}\\
&\quad = -q_{B_1C_2}q_{C_2B_2}q_{B_2B_1}\delta^{C_1}_{B_2}\,\chi^{B_1}_{\ C_2} + q_{C_1B_1}q_{B_2B_1}C_{B_2C_2}\,\chi^{B_1}_{\ C'_1}\\
&\qquad + q_{C_2B_2}q_{B_1C_2}C^{C_1B_1}\,\chi^{B'_2}_{\ C_2} - q_{B_2C_1}\delta^{B_1}_{C_2}\,\chi^{B'_2}_{\ C'_1}\,.
\end{aligned} \tag{6.29}$$

Not all of these functionals are linearly independent because

$$\chi^{B'}_{\ A'} = -q_{AB}\,\chi^{A}_{\ B}. \tag{6.30}$$

From (6.30) we see that a basis of tangent vectors on $SO_{q,r=1}(N+2)$ is given by

$$\{\chi^{A}_{\ B}\quad \text{with } A + B > N+1,\ A, B\!: 0 = \circ, 1, 2, \ldots, N, N+1 = \bullet\}. \tag{6.31}$$

They define a bicovariant differential calculus on $SO_{q,r=1}(N+2)$. The projected bicovariant calculus on $ISO_{q,r=1}(N)$ is therefore characterized by the basis of tangent vectors

$$\chi^a{}_b = \lim_{r\to 1} \frac{1}{\lambda}[f_c{}^{ca}{}_b - \delta^a_b \varepsilon] \quad \text{with } a+b > N+1, \tag{6.32}$$

$$\chi^\bullet{}_b = \lim_{r\to 1} \frac{1}{\lambda} f_\bullet{}^{\bullet\bullet}{}_b, \qquad \chi^\bullet{}_\bullet = \lim_{r\to 1} \frac{1}{\lambda}\, [f_\bullet{}^{\bullet\bullet}{}_\bullet - \varepsilon], \tag{6.33}$$

indeed Theorem 5.4 assures that these functionals annihilate H, while from Note 5.1 it is not difficult to see that the remaining functionals $\chi^a{}_\bullet = (1/\lambda) f_\bullet{}^{\bullet a}{}_\bullet$ do not vanish on H. The q-Lie algebra, in virtue of Theorem 6.2, is a q-Lie subalgebra of $SO_{q,r=1}(N+2)$. It follows that the $\chi^{c_1}{}_{c_2}$, $\chi^{b_1}{}_{b_2}$ q-commutations read as in Eq. (6.29) with lower case indices: they give the $SO_{q,r=1}(N)$ q-Lie algebra. The remaining commutations are (see (6.29)):

$$\chi^{c_1}{}_{c_2}\chi_{b_2} - \frac{q_{c_1\bullet}}{q_{c_2\bullet}} q_{b_2c_1} q_{c_2b_2} \chi_{b_2}\chi^{c_1}{}_{c_2} = \frac{q_{c_1\bullet}}{q_{c_2\bullet}}[C_{b_2c_2}\chi_{c'_1} - \delta^{c_1}_{b_2} q_{c_2c_1}\chi_{c_2}]\,, \tag{6.34}$$

$$\chi_{c_2}\chi_{b_2} - \frac{q_{b_2\bullet}}{q_{c_2\bullet}} q_{c_2b_2}\chi_{b_2}\chi_{c_2} = 0, \tag{6.35}$$

$$\chi^{c_1}{}_{c_2}\chi^\bullet{}_\bullet - \chi^\bullet{}_\bullet\chi^{c_1}{}_{c_2} = 0, \qquad \chi_{c_2}\chi^\bullet{}_\bullet - \chi^\bullet{}_\bullet\chi_{c_2} = -\chi_{c_2}, \tag{6.36}$$

where we have defined $\chi_a \equiv \chi^\bullet{}_a$. The exterior differential reads, $\forall a \in ISO_{q,r}(N)$

$$\mathrm{d}a = (\chi^a{}_b * a)\Omega_a{}^b + (\chi^\bullet{}_b * a)\Omega_\bullet{}^b + (\chi^\bullet{}_\bullet * a)\Omega_\bullet{}^\bullet, \quad a+b > N+1, \tag{6.37}$$

where $\Omega_a{}^b$, $\Omega_\bullet{}^b$, and $\Omega_\bullet{}^\bullet$ are the one-forms dual to the tangent vectors (6.32) and (6.33). Notice that the tangent vectors $\chi^a{}_b$ and χ_b close on the q-Lie algebra (6.34), (6.35) and (6.29) with lower case indices. This suggests a reduction of the bicovariant calculus containing only the $\chi^a{}_b$ and $\chi^\bullet{}_b$ tangent vectors. An explicit formulation, in agreement with [7], is given in [3].

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